9.2 General Fourier Series and Convergence

$f(x)$ period 2π given $-\pi < x < \pi$

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(nx) + b_n \sin(nx)\right]$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

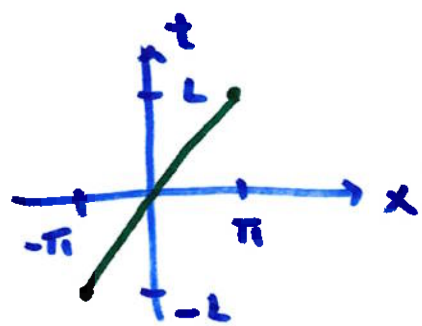
first, let's generalize the period

now $2L$ (L: half period), $-L < t < L$

define ~~x/z~~ ~~x/z~~ $t = \frac{L}{\pi}x$

$x = -\pi \rightarrow t = -L$

$x = \pi \rightarrow t = L$



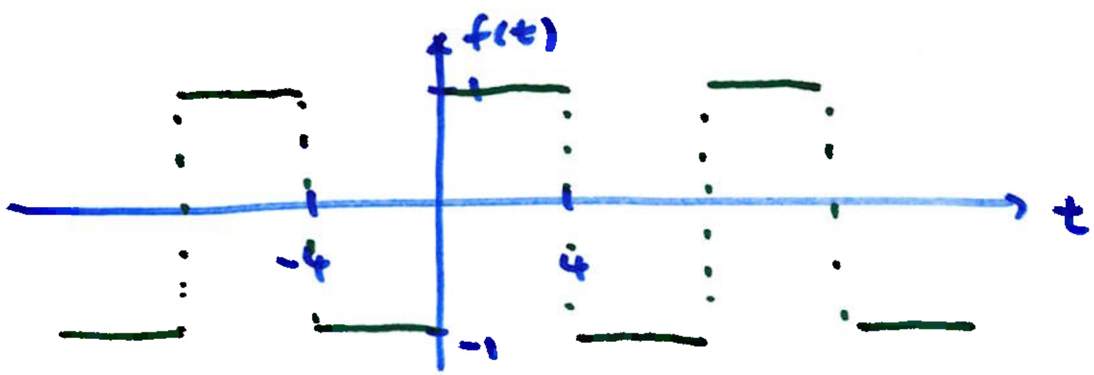
$x = \frac{\pi}{L} t \qquad dx = \frac{\pi}{L} dt$

$$\boxed{f(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{L} t\right) + b_n \sin\left(\frac{n\pi}{L} t\right) \right]}$$

$$a_n = \frac{1}{\pi} \int_{-L}^{L} f(t) \cos\left(\frac{n\pi}{L} t\right) \cdot \frac{\pi}{L} \, dt$$

$$\boxed{\begin{aligned} a_n &= \frac{1}{L} \int_{-L}^{L} f(t) \cos\left(\frac{n\pi t}{L}\right) dt \\ b_n &= \frac{1}{L} \int_{-L}^{L} f(t) \sin\left(\frac{n\pi t}{L}\right) dt \end{aligned}}$$

<u>example</u> $\quad f(t) = \begin{cases} -1 & -4 < t < 0 \\ 1 & 0 < t < 4 \end{cases} \quad$ period 8 $\quad (L = 4)$



$$a_0 = \frac{1}{4}\underbrace{\int_{-4}^{4} f(t)\,dt}_{\text{net area under } f(t)} = 0$$

$$a_n = \frac{1}{4}\int_{-4}^{4} f(t)\cos\left(\frac{n\pi t}{4}\right)dt = \dots = 0$$

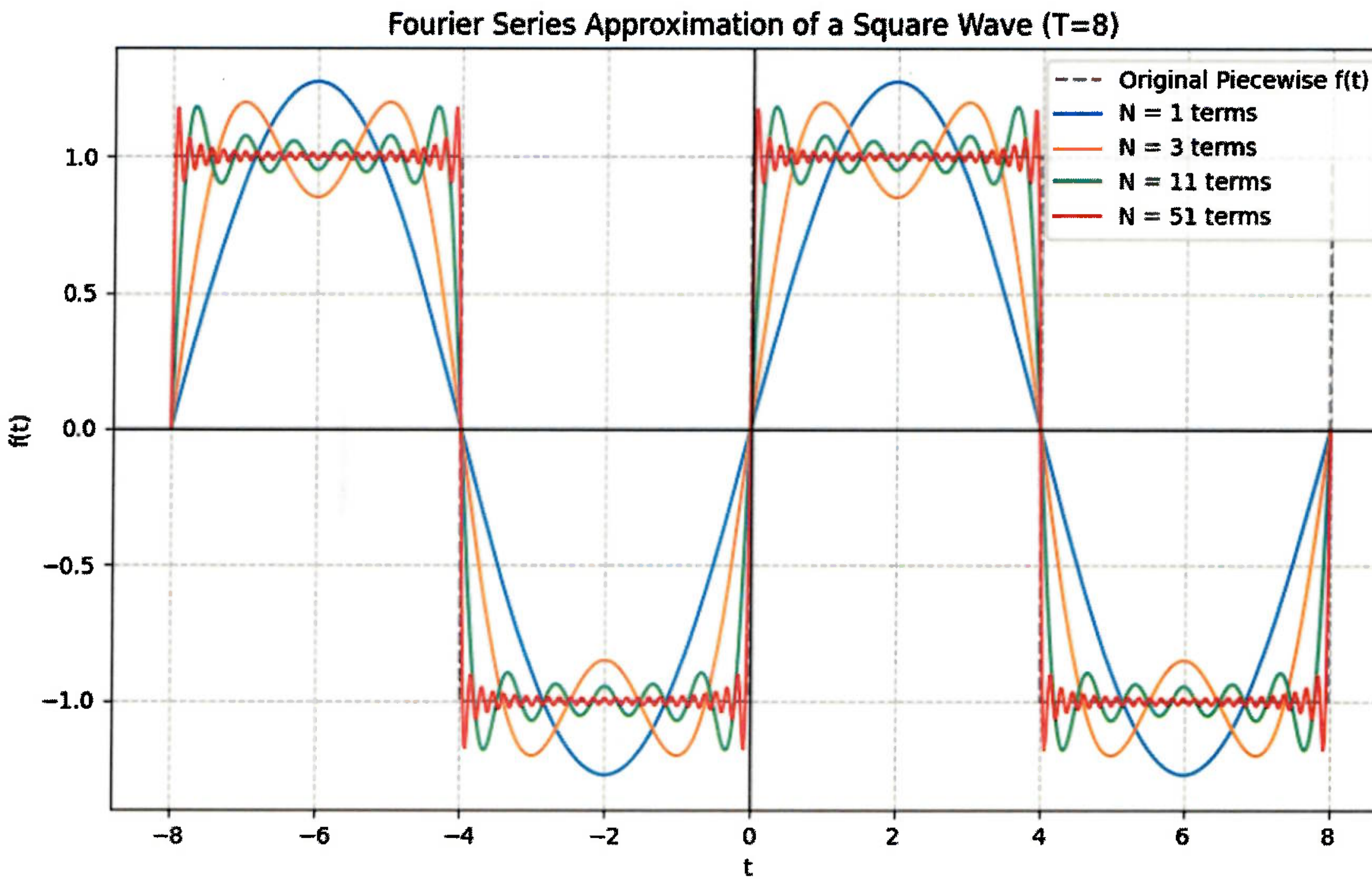
$$b_n = \frac{1}{4}\int_{-4}^{4} f(t)\sin\left(\frac{n\pi t}{4}\right)dt = \dots = \frac{2}{n\pi}\left[1-\cos(n\pi)\right]$$

$$= \frac{2}{n\pi}\left[1-(-1)^n\right] = \begin{cases} \frac{4}{n\pi} & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$$

$$\text{so, } f(t) \sim \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1-(-1)^n\right]\sin\left(\frac{n\pi}{4}t\right)$$

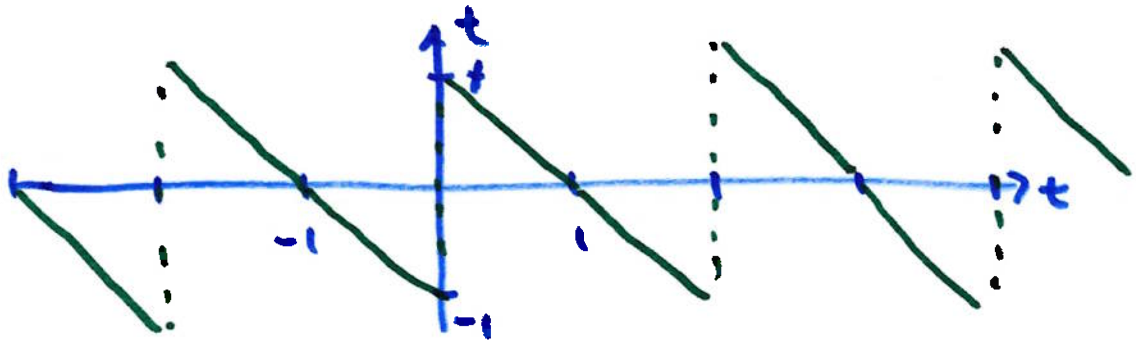
$$\sim \sum_{n \text{ odd}}^{\infty} \frac{4}{n\pi}\sin\left(\frac{n\pi}{4}t\right)$$

$$\frac{4}{\pi}\sin\left(\frac{\pi}{4}t\right) + \frac{4}{3\pi}\sin\left(\frac{3\pi}{4}t\right) + \frac{4}{5\pi}\sin\left(\frac{5\pi}{4}t\right) + \cdots$$


Fourier Series Approximation of a Square Wave (T=8)
Original Piecewise f(t)
N = 1 terms
N = 3 terms
N = 11 terms
N = 51 terms
f(t)
t
1.0
0.5
0.0
−0.5
−1.0
−8
−6
−4
−2
0
2
4
6
8

example

$$f(t) = \begin{cases} -1-t & -1 < t < 0 \\ 1-t & 0 < t < 1 \end{cases} \quad \text{period } 2 \quad (L = 1)$$

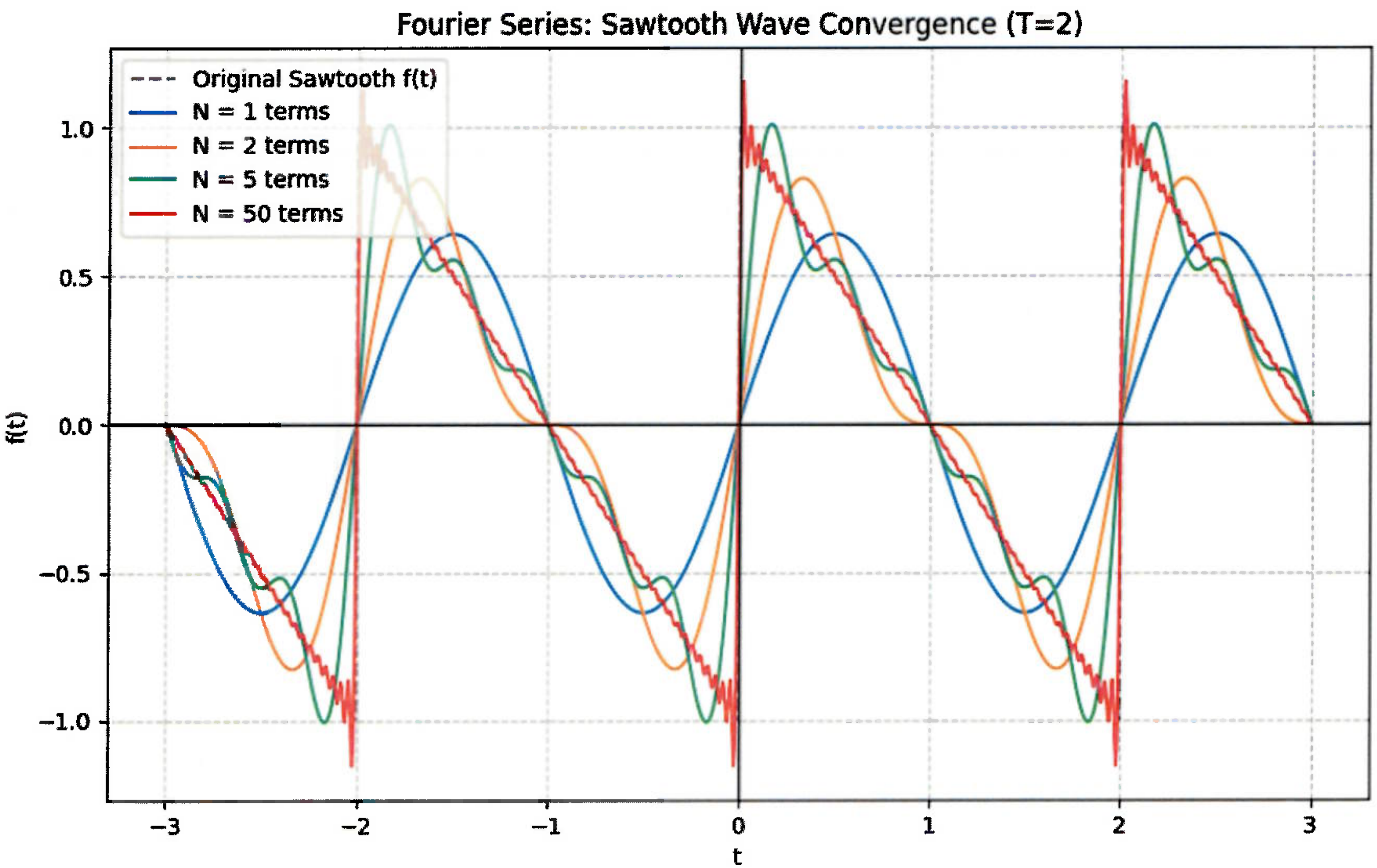


$$a_0 = \frac{1}{1}\int_{-1}^{1} f(t)\,dt = 0$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(t)\cos(n\pi t)\,dt = \cdots = 0$$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(t)\sin(n\pi t)\,dt = \frac{2}{n\pi}$$

$$f(t) \sim \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin(n\pi t) \sim \frac{2}{\pi}\sin(\pi t) + \frac{2}{2\pi}\sin(2\pi t) + \frac{2}{3\pi}\sin(3\pi t) + \cdots$$

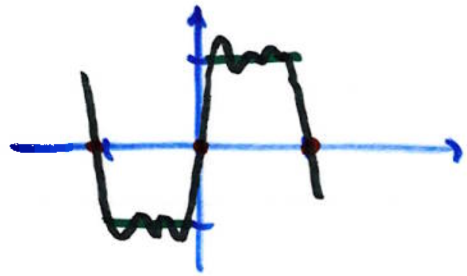

Fourier Series: Sawtooth Wave Convergence (T=2)
Original Sawtooth f(t)
N = 1 terms
N = 2 terms
N = 5 terms
N = 50 terms
1.0
0.5
0.0
−0.5
−1.0
f(t)
−3
−2
−1
0
1
2
3
t

Two main features: at discontinuity, Fourier series goes through the middle (average)

right before discontinuity

right after

$$\frac{f(t^-)+f(t^+)}{2}$$



why? 1st example

$$f(t) \sim \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1-(-1)^n\right] \underbrace{\sin\left(\frac{n\pi t}{4}\right)}^{t=0 \text{ all } \sin\left(\frac{n\pi t}{4}\right)=0}$$

f(t) discontinuous at $t=0$

(−1 right before, 1 right after)

overshoot right before discontinuity

increasing n doesn't make it go away

but pushes it closer to discontinuity

always ~9% of the jump

Gibbs phenomenon

(artifact noticeable as sound or light)